The Golden Gate Bridge, which crosses the entrance to the San Francisco Bay from the Pacific Ocean, is one of the engineering marvels of the United States. When it was built, its 4200 foot (1280 meter) suspension span was the longest in the world. Almost 2 billion vehicles have crossed the Golden Gate Bridge since its opening in 1937. You will use mathematics to describe the parabola formed by the suspension cables on a bridge.
Problem 1   General Form to Standard Form

Write the equation of each conic section in standard form. Then, determine the type of conic section represented by the equation.

1. \[x^2 + y^2 - 4x + 12y + 15 = 0\]

2. \[4x^2 + y^2 - 16 = 0\]
3. \[ 5x^2 + 5y^2 + 10x + 30y - 14 = 0 \]

4. \[ x^2 + 8x + y + 10 = 0 \]

5. \[ x^2 - 4y^2 - 6x - 8y - 59 = 0 \]

6. \[ x^2 + y^2 - 81 = 0 \]
7. \(9x^2 + y^2 - 72x + 135 = 0\)

8. \(25x^2 - 9y^2 + 90y = 0\)

9. \(x^2 - 9y = 0\)

10. \(100x^2 - 49y^2 - 4900 = 0\)
11. \( y^2 - 5x = 0 \)

12. \( 25x^2 + 8y^2 - 100x - 80y + 100 = 0 \)

13. \( y^2 - 2x - 6y + 9 = 0 \)

14. \( x^2 + y^2 + 20x + 19 = 0 \)
15. \(4x^2 - 4y^2 = 36\)

16. \(-4x^2 - 16y^2 + 64 = 0\)

**Problem 2  Analyzing the General Form of Conic Sections**

1. List the equations in general form from Problem 1 that represent circles.

   a. What do you notice about the coefficients of the \(x^2\) and \(y^2\) terms?

   b. Why do you think this relationship occurs between the coefficients of the \(x^2\) and \(y^2\) terms?

   c. What do you notice about the coefficients of the \(x\)- and \(y\)-terms?
d. What affect do the $x$- and $y$-terms have on the circle?

e. The general form of a conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$. Summarize your conclusions from parts (b) and (d) using the variables $A$, $B$, $C$, and $D$.

2. List the equations in general form from Problem 1 that represent ellipses.

a. What do you notice about the coefficients of the $x^2$ and $y^2$ terms?

b. Why do you think this relationship occurs between the coefficients of the $x^2$ and $y^2$ terms?

c. What do you notice about the coefficients of the $x$- and $y$-terms?
d. What affect do the x- and y-terms have on the ellipses?

e. The general form of a conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$. Summarize your conclusions from parts (b) and (d) using the variables $A, B, C,$ and $D$.

3. List the equations in general form from Problem 1 that represent hyperbolas.

a. What do you notice about the coefficients of the $x^2$ and $y^2$ terms?

b. Why do you think this relationship occurs between the coefficients of the $x^2$ and $y^2$ terms?

c. What do you notice about the coefficients of the x- and y-terms?
d. What affect do the $x$- and $y$-terms have on the hyperbola?

e. The general form of a conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$. Summarize your conclusions from parts (b) and (d) using the variables $A$, $B$, $C$, and $D$.

4. List the equations in general form from Problem 1 that represent parabolas.

a. What do you notice about the coefficients of the $x^2$ and $y^2$ terms?

b. Why do you think this relationship occurs between the coefficients of the $x^2$ and $y^2$ terms?

c. What do you notice about the coefficients of the $x$- and $y$-terms?
d. What affect do the $x$- and $y$-terms have on the ellipses?

e. The general form of a conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$.
Summarize your conclusions from parts (b) and (d) using the variables $A, B, C,$ and $D$. 
5. Complete the following table.

<table>
<thead>
<tr>
<th>Conic Section</th>
<th>Relationship Between $A$ and $B$</th>
<th>Inclusion of $C$ and $D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td></td>
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</tr>
<tr>
<td>Ellipse</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hyperbola</td>
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<tr>
<td>Parabola</td>
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</tr>
</tbody>
</table>

General Form of a Conic Section: $Ax^2 + By^2 + Cx + Dy + E = 0$
Problem 3 Determining the Conic Section from its General Form

Determine the type of conic section represented by each equation.

1. \(x^2 + 6x + 8y + 17 = 0\)

2. \(x^2 - 4y^2 + 4x + 32y - 96 = 0\)

3. \(7x^2 + 7y^2 = 175\)

4. \(9x^2 + 16y^2 + 36x - 96y + 36 = 0\)

5. \(x^2 + y^2 + 8x - 4y - 5 = 0\)

6. \(7x^2 - 7y^2 = 175\)

Be prepared to share your methods and solutions.
### Problem 1  Key Characteristics of Conics

Determine the type of conic section represented by each equation. Then, write the equation in standard form and identify the key characteristics of the conic. Include the following key characteristics. Finally, graph and label the conic.

<table>
<thead>
<tr>
<th>Key Characteristics</th>
<th>Circle</th>
<th>Ellipse</th>
<th>Hyperbola</th>
<th>Parabola</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>• center</td>
<td>• center</td>
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<td>• vertex</td>
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<td></td>
<td>• radius</td>
<td>• vertices</td>
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<td>• axis of symmetry</td>
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<td>• co-vertices</td>
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<td>• directrix</td>
</tr>
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<td></td>
<td>• eccentricity</td>
<td>• eccentricity</td>
<td>• concavity</td>
</tr>
</tbody>
</table>
1. \( 169x^2 + 25y^2 + 350y - 3000 = 0 \)
2. \(64x^2 - 36y^2 + 72y - 2340 = 0\)
3. \( x^2 + y^2 + 4x + 2y - 4 = 0 \)
4. $y^2 - 12x - 8y + 124 = 0$
5. \[64x^2 - 225y^2 + 384x + 2250y + 9351 = 0\]
6. \[4x^2 + 4y^2 - 48x - 40y + 144 = 0\]
7. \[2x^2 + 32x + 28y + 128 = 0\]
8. \(16x^2 + 25y^2 - 96x - 300y + 644 = 0\)

Be prepared to share your methods and solutions.
1. The main cables of a suspension bridge are parabolic. The parabolic shape allows the cables to bear the weight of the bridge evenly. The distance between the towers is 900 feet and the height of each tower is about 75 feet.

Write an equation for the parabola representing the cable between the two towers.
2. The cross section of a satellite dish is a parabola. The satellite dish is 5 feet wide at its opening and 1 foot deep. The receiver of the satellite dish should be placed at the focus of the parabola. How far should the receiver be placed from the vertex of the satellite dish?
3. Many carnivals and amusement parks have mirrors that are parabolic. When you look at your reflection in a parabolic mirror, your image appears distorted and makes you look taller or shorter depending on the shape of the mirror. The focal length of a mirror is the distance from the vertex to the focus of the mirror. Consider a mirror that is 72 inches tall with a vertex that is concave 6 inches from the top and bottom edges of the mirror. What is the focal length of the mirror?
4. The orbit of the Earth follows a path that is in the shape of an ellipse with an eccentricity of \( \frac{1}{60} \). The length of the major axis is 186 million miles. Let the Sun represent one focus of the ellipse. Determine the minimum and maximum distance from the Earth to the Sun.
5. The orbit of the Moon follows a path that is in the shape of an ellipse with the Earth at one focus. The minimum distance from the Moon to the Earth is 221,642 miles. The maximum distance from the Moon to the Earth is 252,710 miles. Write an equation representing the Moon’s elliptical orbit around the Earth.
6. The underpass of a bridge is in the shape of half of an ellipse and is 100 feet wide and 23 feet high. A rectangular barge is 60 feet wide and 20 feet above the water. Can the barge pass under the bridge?
7. Cooling towers for nuclear power plants are hyperbolic. The shape of these towers is formed by rotating a hyperbola around a vertical axis to form a three-dimensional shape called a hyperboloid. Typically, these cooling towers are slightly larger on the bottom than the top. The design of a hyperboloid creates a draft bringing cool air into the system to aid in the cooling process.

Consider the following hyperboloid. Each coordinate is measured in feet.

\[ (155, 123) \]
\[ (-155, 123) \]
\[ (230, -442) \]
\[ (-230, -442) \]

Verte (147, 0)

\[ y \]
\[ x \]

a. What is the equation for the hyperbola used to generate this three-dimensional cooling tower?
b. The width of the cooling tower at the base is 150 feet wider than it is at the top. How tall is the cooling tower?
8. Radio signals submitted from a transmitter form a pattern of concentric circles. Use the coordinate plane to graph the broadcast range of two radio stations and the location of a radio listener’s home.

a. Radio station WPOP plays top 40 hits. Plot the location of station WPOP as a point at the origin.

b. Sal lives 24 miles north and 32 miles east of the radio station. His home is located on the edge of WPOP’s broadcast range. Plot the location of Sal’s home as a point on the grid. How far must a radio signal travel to reach Sal’s home?

c. Write an equation of a circle to represent station WPOP’s maximum listening area. Graph the equation.
d. A new station, \( WREQ \), just starting broadcasting. It is located 50 miles east of \( WPOP \). Plot the location of station \( WREQ \) as a point on the grid.

e. \( WREQ \)'s signal travels 30 miles. Will Sal be able to listen to \( WREQ \) from his home?

f. Write an equation of a circle to represent station \( WREQ \)'s maximum listening area. Graph the equation.
g. Show that a point of intersection of the two radio stations’ maximum listening areas is Sal’s home using the equations of circles representing both radio stations.

Be prepared to share your methods and solutions.
Previously, you examined the cross sections formed by the intersection of a plane and a double-napped cone. These cross sections are the conic sections, specifically a circle, an ellipse, a hyperbola, and a parabola. In addition, there are three degenerate conics—a point, a line, and intersecting lines.

A sphere is the set of all points in three-dimensional space equidistant from a fixed point called the center. The radius of a sphere is the distance from a point on the sphere to its center.

Consider the intersection of a sphere and a plane.
1. Consider a sphere. When a plane intersects a sphere so that the plane passes through the center, a circle is formed. How does the diameter of the circle compare to the diameter of the sphere? Use a complete sentence in your answer.

A great circle is a cross section of a sphere and a plane that passes through the center of the sphere. The center of a great circle is the same as the center of the sphere. The radius of a great circle is equal to the radius of the sphere.

2. Can a different plane intersect the sphere so that another great circle is formed?

3. How many great circles can you form by intersecting a plane and a sphere?

4. Is every intersection of a plane and a sphere a great circle? Use a complete sentence to explain your reasoning.

5. Suppose that a plane intersects the sphere so that the plane does not pass through the center. How does the diameter of this circle compare to the diameter of the sphere? Use a complete sentence to explain your reasoning.
6. Do all of the intersections of a plane with a sphere create circles? If not, describe a situation in which the intersection of a plane with a sphere is not a circle. Use a complete sentence in your answer.

Problem 2  Equation of a Sphere

1. How is the definition of a sphere similar to the definition of a circle?

2. How is the definition of a sphere different than the definition of a circle?

3. Consider the set of all points in the coordinate plane that are four units from the origin.
   a. Write an equation to represent the set of all points \((x, y)\) that have a distance of four units from the origin.

   b. Describe how to convert the equation in part (a) to the equation of a circle.
4. Consider the set of all points in the three-dimensional coordinate system that have a distance of four units from the origin.

a. Write an equation to represent the set of all points \((x, y, z)\) that have a distance of four units from the origin.

\[(0, 0, 0)\]

\[(x, y, z)\]

b. Describe how to convert the equation in part (a) to the equation of a sphere.
The **standard form of a sphere** centered at the origin with radius \( r \) is 
\[
x^2 + y^2 + z^2 = r^2.
\]

5. Consider the set of all points in the coordinate plane that are five units from the point \((-1, 3)\).

a. Write an equation to represent the set of all points \((x, y)\) that are five units from the point \((-1, 3)\).

\[
(x - (-1))^2 + (y - 3)^2 = 5^2.
\]

b. Describe how to convert the equation in part (a) to the equation of a circle.
6. Consider the set of all points in the three-dimensional coordinate system that are five units from the point \((-1, 3, 4)\).

\[ (x, y, z) \] \((-1, 3, 4) \]

a. Write an equation to represent the set of all points \((x, y, z)\) that are five units from the point \((-1, 3, 4)\).

b. Describe how to convert the equation in part (a) to the equation of a sphere.

The standard form of a sphere centered at the point \((h, k, j)\) with radius \(r\) is

\[ (x - h)^2 + (y - k)^2 + (z - j)^2 = r^2. \]
Problem 3  Center and Radius of a Sphere

1. Write an equation in standard form of each sphere.
   a. A sphere with a radius of 10 and center (3, −7, 5).
   b. A sphere with a radius of 3 and center (4, 0, −9).
   c. A sphere with a radius of 1 and center (0, 0, −2).
   d. A sphere with a radius of 4.5 and center (−8, 1, −5).
   e. A sphere with a radius of 12 and center (6, −6, 0).

2. Determine the center and radius of each sphere.
   a. A sphere represented by the equation \((x − 3)^2 + (y + 4)^2 + (z − 7)^2 = 81\).
   b. A sphere represented by the equation \((x − 8)^2 + (y + 11)^2 + z^2 = 225\).
   c. A sphere represented by the equation \((x + 1.5)^2 + y^2 + (z − 1.5)^2 = 49\).
   d. A sphere represented by the equation \(x^2 + (y + 2)^2 + z^2 = 9\).
   e. A sphere represented by the equation \((x + 9)^2 + (y − 11)^2 + (z + 6)^2 = 20\).

3. Write an equation in standard form of each sphere. Then, determine the center and radius of the sphere.
   a. \(x^2 + y^2 + z^2 + 6x + 16y − 4z + 61 = 0\)
b. \(x^2 + y^2 + z^2 - 12y - 2z + 12 = 0\)

c. \(3x^2 + 3y^2 + 3z^2 + 6x - 6y - 21 = 0\)

d. \(2x^2 + 2y^2 + 2z^2 + 12x - 16y - 20z + 2 = 0\)

e. \(5x^2 + 5y^2 + 5z^2 + 70x + 40y + 90z + 630 = 0\)
Problem 4  Intersection of a Sphere Centered at the Origin and a Plane

1. Consider the equation \( x^2 + y^2 = 49 \) in the three-dimensional coordinate system.
   a. What is the center and radius of the circle in the \( xy \)-plane?
   
   b. What is the center and radius of the circle in the plane \( z = 2 \)?
   
   c. What is the center and radius of the circle in the plane \( z = -5 \)?
   
   d. Describe the graph of \( x^2 + y^2 = 49 \) in the three-dimensional coordinate system.

2. Consider the equation \( x^2 + z^2 = 49 \) in the three-dimensional coordinate system.
   a. What is the center and radius of the circle in the \( xz \)-plane?
   
   b. What is the center and radius of the circle in the plane \( y = 2 \)?
   
   c. What is the center and radius of the circle in the plane \( y = -5 \)?
   
   d. Describe the graph of \( x^2 + z^2 = 49 \) in the three-dimensional coordinate system.
3. Consider the equation \( y^2 + z^2 = 49 \) in the three-dimensional coordinate system.

   a. What is the center and radius of the circle in the \( yz \)-plane?

   b. What is the center and radius of the circle in the plane \( x = 2 \)?

   c. What is the center and radius of the circle in the plane \( x = -5 \)?

   d. Describe the graph of \( y^2 + z^2 = 49 \) in the three-dimensional coordinate system.

4. Consider the sphere \( x^2 + y^2 + z^2 = 25 \) and the plane \( x = 3 \).

   a. What is the center and radius of the sphere?

   b. Describe the graph of the plane \( x = 3 \) in the three-dimensional coordinate system.

   c. Substitute the plane \( x = 3 \) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.
d. Is the intersection a point, a circle, or a great circle? How do you know?

5. Consider the sphere \( x^2 + y^2 + z^2 = 25 \) and the plane \( z = 4 \).
   a. Substitute the plane \( z = 4 \) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.
   
   b. Is the intersection a point, a circle, or a great circle? How do you know?

6. Consider the sphere \( x^2 + y^2 + z^2 = 25 \) and the plane \( y = 5 \).
   a. Substitute the plane \( y = 5 \) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.
   
   b. Is the intersection a point, a circle, or a great circle? How do you know?
7. Consider the sphere \( x^2 + y^2 + z^2 = 25 \) and the plane \( x = 0 \).
   
   **a.** Substitute the plane \( x = 0 \) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.

   **b.** Is the intersection a point, a circle, or a great circle? How do you know?

8. Consider the sphere \( x^2 + y^2 + z^2 = 25 \) and the plane \( z = 10 \).
   
   **a.** Substitute the plane \( z = 10 \) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.

   **b.** Is the intersection a point, a circle, or a great circle? How do you know?

9. Describe the values of \( a \) in the equation of a plane \( x = a, y = a, \) or \( z = a \) that result in each intersection of a sphere centered at the origin and a plane.
   
   **a.** A great circle
Problem 5  Intersection of a Sphere Not Centered at the Origin and a Plane

1. Consider the sphere \((x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 169\) and the plane \(z = 9\).

a. Substitute the plane \(z = 9\) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.

b. Is the intersection a point, a circle, or a great circle? How do you know?

c. A circle that is not a great circle

d. No intersection
2. Consider the sphere \((x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 169\) and the plane \(x = 2\).
   
   a. Substitute the plane \(x = 2\) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.

   b. Is the intersection a point, a circle, or a great circle? How do you know?

3. Consider the sphere \((x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 169\) and the plane \(y = 10\).
   
   a. Substitute the plane \(y = 10\) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.

   b. Is the intersection a point, a circle, or a great circle? How do you know?
4. Consider the sphere \((x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 169\) and the plane \(z = -10\).
   a. Substitute the plane \(z = -10\) into the equation of the sphere. Write the equation with all variables on one side of the equation and the constants on the other side of the equation.

   b. Is the intersection a point, a circle, or a great circle? How do you know?

5. Describe the values of \(a\) in the equation of a plane \(x = a, y = a, \) or \(z = a\) that results in each intersection of the sphere \((x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 169\) and a plane.
   a. A great circle
   b. A point
c. A circle that is not a great circle

d. No intersection

Problem 6 Intersection of a Sphere and a Plane

Determine whether the intersection of each sphere and plane is a point, a circle, or a great circle. If the intersection is a circle or a great circle, identify the equation of the circle and its center and radius. If the intersection is a point, identify the coordinates of the point.

1. \( x^2 + y^2 + z^2 = 625 \)
   
   \( z = 7 \)
2. \((x - 2)^2 + (y + 1)^2 + (z - 8)^2 = 225\)
   
   \[
x = 17
   \]

3. \((x + 3)^2 + (y + 5)^2 + (z + 7)^2 = 676\)
   
   \[
y = -5
   \]

4. \((x + 8)^2 + y^2 + (z - 3)^2 = 100\)
   
   \[
x = 0
   \]

5. \((x - 2)^2 + (y + 4)^2 + z^2 = 400\)
   
   \[
y = -24
   \]

Be prepared to share your methods and solutions.