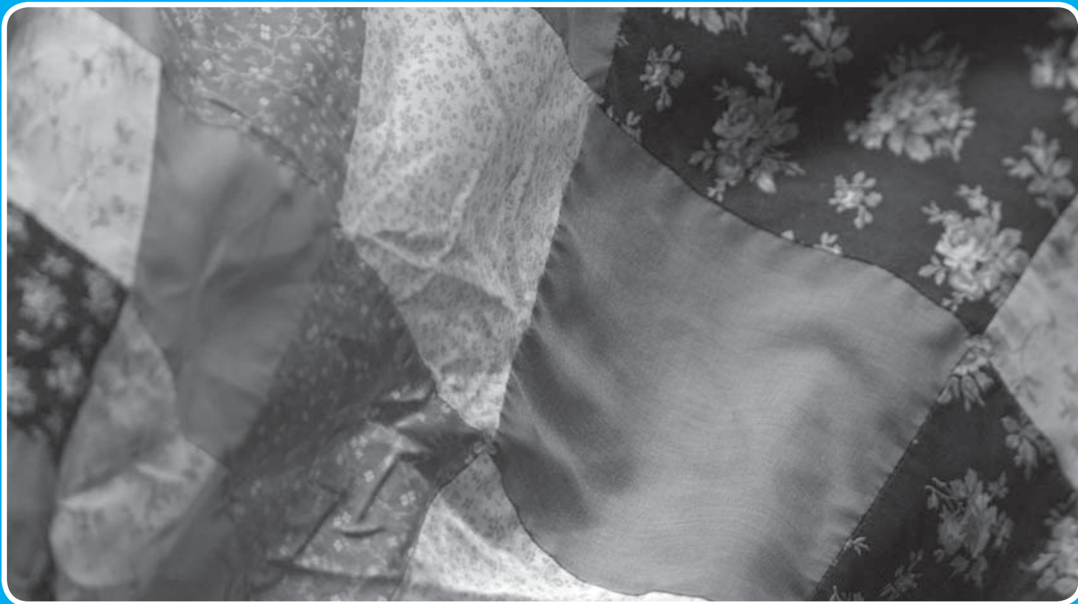
CHAPTER

6 Properties of Quadrilaterals



The earliest evidence of quilting is an ivory carving from the 35th century BC. It shows the king of the Egyptian First Dynasty wearing a quilted cloak. You will examine quilts formed by using tessellations.

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6.1 Quilting and Tessellations
Introduction to Quadrilaterals

Objectives

In this lesson, you will:

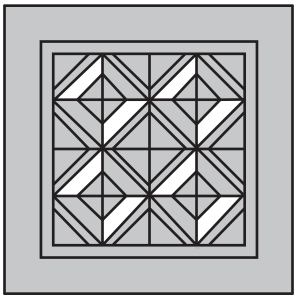
- Classify quadrilaterals.
- Name quadrilaterals and parts of quadrilaterals.
- Draw a Venn diagram that shows the relationships among quadrilaterals.

Key Terms

- tessellation
- parallelogram
- rhombus
- rectangle
- square
- trapezoid
- kite
- Venn diagram

Quilts are often made of repeating geometric shapes that form *tessellations*. A **tessellation** of a plane is a collection of polygons that are arranged so that they cover the plane with no holes or gaps.

Some quilts are created in a block pattern, such as the one shown. Copies of these blocks are created by sewing different patterns or colors together. Then the blocks are sewn together to form the quilt.



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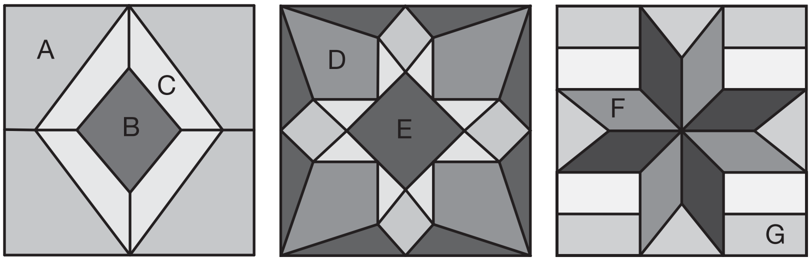
Problem 1 Describing Quadrilaterals

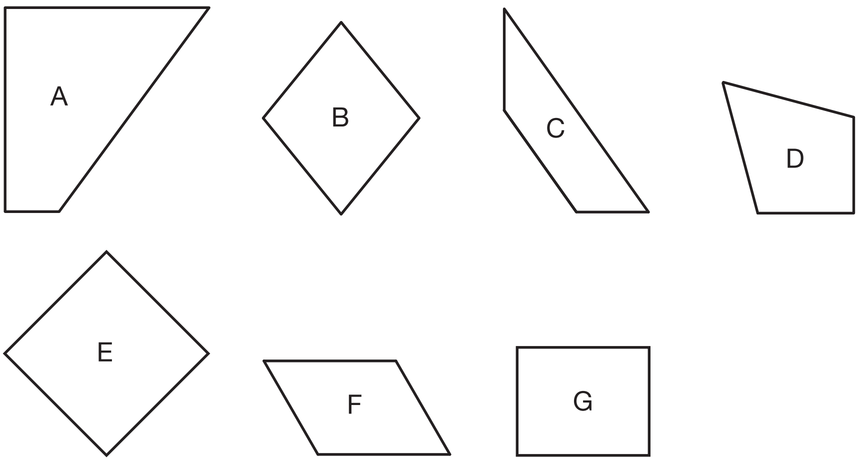


A. Shapes A through G labeled in these quilt squares are polygons. What is the classification for these polygons by the number of sides in the polygon?

B. How are these polygons the same? How are they different?

Investigate Problem 1

Use these quadrilaterals to answer the following questions.



1. For each quadrilateral, use your protractor to determine which interior angles are right angles. Mark these angles as right angles on the quadrilaterals.
2. For each quadrilateral, use your protractor to determine which interior angles are congruent. Mark the congruent angles on the quadrilaterals.

3. For each quadrilateral, determine which sides are congruent. Mark the congruent sides on the quadrilaterals.

4. For each quadrilateral, determine which sides are parallel. Mark the parallel sides on the quadrilaterals.

5. Name the quadrilaterals in which both pairs of opposite sides are parallel.

These quadrilaterals are **parallelograms.**

6. Name the quadrilateral(s) in which both pairs of opposite sides are parallel and all the sides are congruent.

This type of quadrilateral is called a **rhombus**. The plural form of rhombus is *rhombi*.

7. Name the quadrilateral(s) in which both pairs of opposite sides are parallel and the interior angles are right angles.

These quadrilaterals are **rectangles.**

8. Name the quadrilateral(s) in which both pairs of opposite sides are parallel, the interior angles are right angles, and the sides are congruent.

These quadrilaterals are **squares.**

9. Name the quadrilateral(s) in which only one pair of opposite sides are parallel.

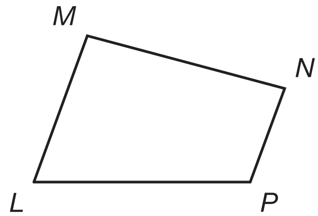
These quadrilaterals are **trapezoids.**

10. Which of the quadrilaterals has yet to be classified by its sides or angles?

Describe this quadrilateral in terms of its sides.

This quadrilateral is a *kite*. A **kite** is a quadrilateral in which two pairs of adjacent sides are congruent, but the opposite sides are not congruent.

11. Quadrilaterals are named by their vertices. For instance, this quadrilateral can be named quadrilateral *LMNP*, quadrilateral *MLPN*, but not quadrilateral *NLMP*.



What does this tell you about how a quadrilateral must be named?

12. Draw quadrilateral *WXYZ* so that the quadrilateral is a parallelogram that is not a rectangle. Then name the pairs of parallel sides. Name any congruent angles.

13. Decide whether the following statements are true or false. Explain your reasoning.

All rectangles are squares.

All squares are rectangles.

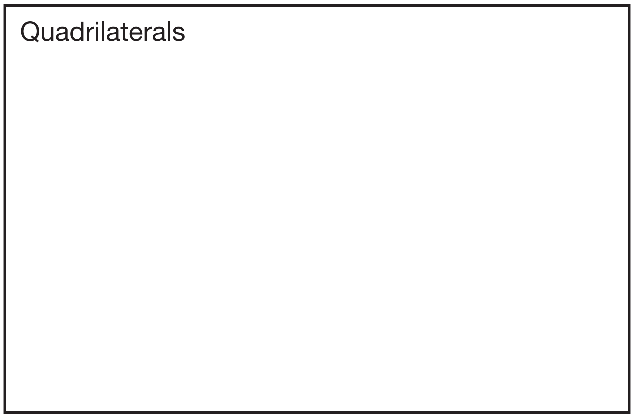
All trapezoids are parallelograms.

All rectangles are parallelograms.

All squares are rhombi.

All quadrilaterals are parallelograms.

14. You can use a **Venn diagram** to show the relationship between different kinds of quadrilaterals.



First, inside the rectangle, draw a large circle that represents all parallelograms.

Now add a circle to the diagram that represents all rhombi. If no rhombus is a parallelogram, then draw the circle that represents the rhombi so that it is outside of the circle that represents parallelograms. If every rhombus is a parallelogram, draw the circle that represents the rhombi inside the circle that represents parallelograms. If some, but not all rhombi are parallelograms, draw the circle that represents the rhombi so that it intersects the circle that represents parallelograms.

Draw the circle that represents all rectangles. Then draw a circle that represents all kites. Then draw a circle that represents all trapezoids.

Complete the Venn diagram by labeling the part of the diagram that represents all squares.

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6.2 When Trapezoids Are Kites
Kites and Trapezoids

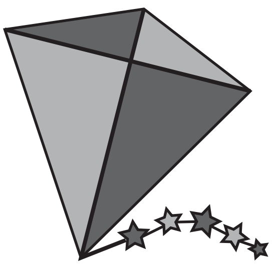
Objectives
In this lesson, you will:
- Determine properties of kites.
- Determine properties of isosceles trapezoids.

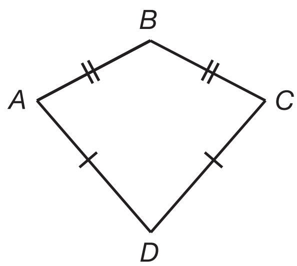
Key Terms
- diagonal
- isosceles trapezoid
- base
- base angles

Simple kites are often in the shapes of simple geometric forms. A diamond kite is in the shape of a (geometric) kite, and a delta conyne kite is in the shape of a trapezoid.

Problem 1 Diamond Kites

A diamond kite and the outline of the kite are shown.





A. Draw a *diagonal* of the kite from vertex *B* to vertex *D*. A **diagonal** is a line segment that is drawn from a vertex to a non-adjacent vertex.

B. What is the relationship between ΔABD and ΔCBD? Explain your reasoning.

C. Are $\angle A$ and $\angle C$ congruent? Explain your reasoning.

D. Can you determine whether $\angle B$ and $\angle D$ are congruent without measuring the angles? Explain your reasoning.

E. What do you know about $\angle ABD$ and $\angle CBD$? What do you know about $\angle ADB$ and $\angle CDB$? Explain your reasoning.

F. What does part (E) tell you about $\overline{BD}$?

G. Suppose that $\angle B$ and $\angle D$ are congruent. Then how does $m\angle ABD$ compare to $m\angle ADB$? Explain your reasoning.

Because $m\angle ABD = m\angle ADB$, what kind of triangle is ΔABD? Explain.

How does AB compare to AD? Explain.

Is this possible? Why or why not?

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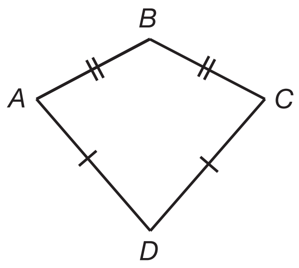
H. Complete the following statement:

If a quadrilateral is a kite, then only __________ of __________ angles are congruent.

Investigate Problem 1

1. The outline of the diamond kite is again shown. Draw both diagonals of the kite on the figure, and label the point of intersection as point *E*.



From Problem 1, we know that $\Delta ABD \cong \Delta CBD$. We also know that $\overline{BD}$ bisects $\angle B$. What does this tell you about the relationship between ΔABE and ΔCBE? Explain your reasoning.

What do you know about the relationship between $\angle AEB$ and $\angle CEB$? Explain your reasoning.

Complete the following statement:

Angle *AEB* and $\angle CEB$ form a _______ pair.

What can you conclude about $m\angle AEB$ and $m\angle CEB$? Explain your reasoning.

Complete the following statement:

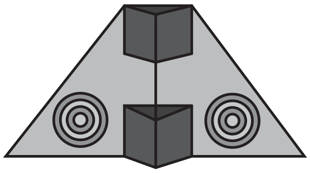
The diagonals of a kite are _______________.

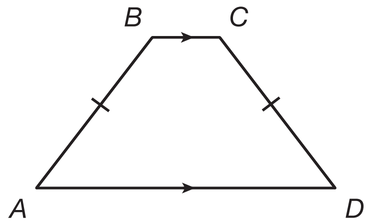
2. Consider $\overline{AC}$, the diagonal that connects the vertices whose angle measures are congruent. Where does $\overline{BD}$ intersect $\overline{AC}$? How do you know? Explain your reasoning.

What relationship does this give between $\overline{BD}$ and $\overline{AC}$?

Problem 2 Delta Conyne Kites

A delta conyne kite and the outline of the kite are shown.





A. What do you notice about the nonparallel sides of the trapezoid?

This trapezoid is called an **isosceles trapezoid.**

B. How do you think $m\angle A$ compares to $m\angle D$? How do you think $m\angle B$ compares to $m\angle C$? Explain your reasoning.

Use a protractor to verify your answers.

C. Recall that the parallel sides of a trapezoid are the **bases** of the trapezoid. The pair of angles that share a base as a side are called a pair of **base angles.**

Name the pairs of base angles in the trapezoid.

Complete the following statement:

The base angles of an isosceles trapezoid are ____________.

Label this information on the figure.

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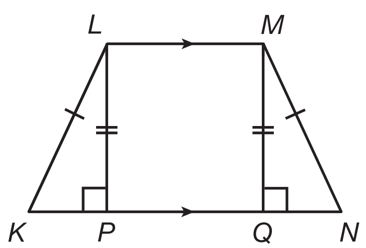
Investigate Problem 2

1. Draw the diagonals of the trapezoid on the figure in Problem 2. Then sketch ΔABC and ΔDCB separately in the space provided. Mark any information on your sketch that you know about the relationships between angles and sides of the triangles.

What can you conclude about the triangles? Explain your reasoning.

Write a statement that tells what you know about the lengths of the diagonals of an isosceles trapezoid.

2. Complete this paragraph proof that shows that the base angles of an isosceles trapezoid are congruent.



We are given that $\overline{KL} \cong$ _____ and _____ $\parallel \overline{KN}$. First draw perpendicular line segments from vertex L and vertex M to ____ to form $\overline{LP}$ and $\overline{MQ}$. Segment $\overline{LP}$ and $\overline{MQ}$ are ____________ because $\overline{LM} \parallel \overline{KN}$, and the distance between two parallel lines is the same from any point on either line. Angle KPL and $\angle NQM$ are right angles because $\overline{LP} \perp$ ____ and $\overline{MQ} \perp$ ____. So ΔKLP and ΔNMQ are ______ triangles with a pair of congruent legs and congruent hypotenuses. By the _________________ Theorem, $\Delta KLP \cong NMQ$. Because $\angle K$ and $\angle N$ are ________________angles of congruent triangles, the angles are congruent. Angle L and $\angle M$ can be shown to be congruent in a similar way.

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6.3 Binocular Stand Design

Parallelograms and Rhombi

Objectives

In this lesson, you will:

- Determine properties of parallelograms.
- Determine properties of rhombi.

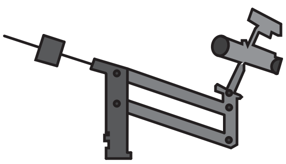
Key Terms

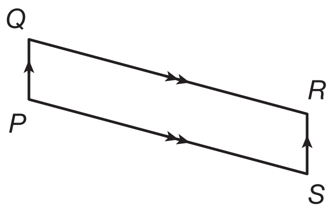
- opposite sides
- consecutive sides
- consecutive angles
- opposite angles

Sometimes, binoculars are better for viewing stars than telescopes. Because it is not reasonable for a person to hold the binoculars for an extended period of time, there are binocular stands that can be used to hold the binoculars. Part of the structure for this stand is in the shape of a parallelogram.

Problem 1 Holding It Steady

A typical binocular stand and an outline of the parallelogram part of the stand are shown.





A. Two sides of a parallelogram that do not intersect are **opposite sides.** Name the pairs of opposite sides in parallelogram *PQRS*.

B. Two sides of a parallelogram that intersect are **consecutive sides.** Name the pairs of consecutive sides in parallelogram *PQRS*.

C. Two angles of a parallelogram that have a side in common are **consecutive angles.** Name the pairs of consecutive angles in parallelogram *PQRS*.

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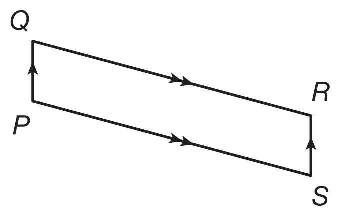
D. Two angles of a parallelogram that do not have a side in common are **opposite angles.** Name the pairs of opposite angles in parallelogram *PQRS*.

E. What do you think is the relationship between the opposite sides of a parallelogram? What do you think is the relationship between the opposite angles of a parallelogram? What do you think is the relationship between the consecutive angles of a parallelogram?

Investigate Problem 1

1. The parallelogram from Problem 1 is shown. Draw the diagonal that connects vertices *P* and *R*.



How does $m\angle PRS$ compare to $m\angle RPQ$? How does $m\angle QRP$ compare to $m\angle SPR$? Explain your reasoning.

What can you conclude about ΔQRP and ΔSPR? Explain your reasoning.

What can you conclude about the opposite sides of a parallelogram? Explain your reasoning.

What can you conclude about $\angle Q$ and $\angle S$? Explain your reasoning.

Can you draw a conclusion about $\angle R$ and $\angle S$? If so, what is your conclusion? Explain your reasoning.

Which other pairs of angles have this same relationship?

Are $\angle P$ and $\angle R$ congruent? Explain your reasoning.

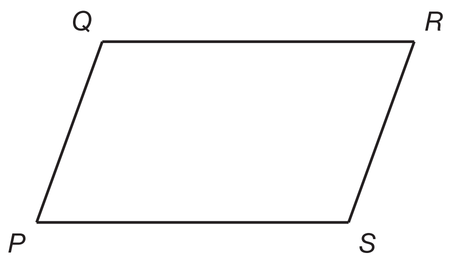
2. Complete the following statements:

The opposite sides of a parallelogram are ___________.

The opposite angles of a parallelogram are ___________.

The consecutive angles in a parallelogram are _______________________.

3. Draw the diagonals on this parallelogram. Label the intersection point as point T. What do you know about the opposite sides of the parallelogram? Label this information on your figure.



Use what you learned in Question 1 to name the congruent triangles formed by the parallelogram and a diagonal.

On the figure, label the congruent angles of the congruent triangles you named. What can you conclude about ΔQTR and ΔSTP? Explain your reasoning.

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What can you conclude about ΔQTP and ΔSTR? Explain your reasoning.

What can you conclude about $\overline{PT}$, $\overline{RT}$, $\overline{QT}$, and $\overline{ST}$? Explain your reasoning.

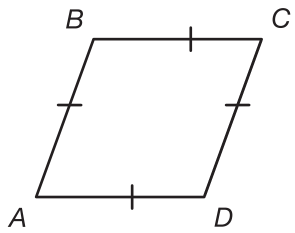
Complete the following statement:

The diagonals of a parallelogram ________ each other.

4. Consider this rhombus. Draw a diagonal that connects vertices A and C.

Take Note

Because a rhombus is a parallelogram, the properties of parallelograms are true for rhombi.



What do you know about ΔABC and ΔADC? Explain your reasoning.

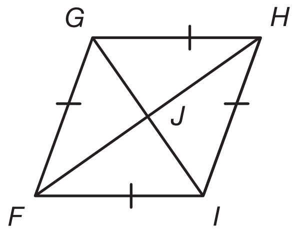
What can you conclude about $\angle BAC$ and $\angle DAC$? What can you conclude about $\angle BCA$ and $\angle DCA$? Explain your reasoning.

How does $\overline{AC}$ relate to $\angle A$? How does $\overline{AC}$ relate to $\angle C$?

Complete the following statement:

The diagonal of a rhombus ________ a pair of opposite angles.

5. Consider this rhombus and its diagonals.



Triangle *GHJ* is congruent to Δ*IHJ*. Why?

How do $m\angle HJG$ and $m\angle HJI$ compare?

Triangle *GJF* is congruent to Δ*IJF*. Why?

How do $m\angle GJF$ and $m\angle IJF$ compare?

Triangle *GHJ* is congruent to Δ*GFJ*. Why?

How do $m\angle HJG$ and $m\angle GJF$ compare?

What is the relationship between $\angle HJG$, $\angle HJI$, $\angle IJF$, and $\angle GJF$? Explain your reasoning.

What does this tell you about the measures of the angles formed by the intersection of the diagonals? Explain your reasoning.

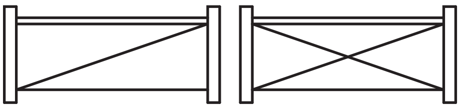
Complete the following statement:

The diagonals of a rhombus are _______________.

6.4 Positive Reinforcement
Rectangles and Squares

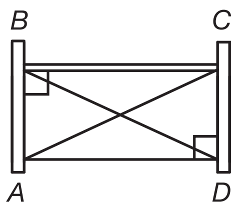
Objectives
In this lesson, you will:
- Determine properties of rectangles and squares.

Fences built to keep livestock in enclosed areas are often built in rectangular sections. These sections are made stronger by adding one or two diagonal wire braces as shown.



Problem 1 Making the Fence Stronger

A section of rectangular fence with two diagonal braces is shown.



A. A rancher is building a section of fence that is 8 feet long and 5 feet tall between two fence posts. Label this information on the figure. About how much wire does the rancher need for each diagonal brace? Show all your work. Round your answer to the nearest tenth, if necessary.

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B. What does part (A) tell you about the diagonals of rectangle *ABCD*? Is this true for all rectangles? Explain your reasoning.

C. Complete the following statement:

The diagonals of a rectangle are ___________.

Investigate Problem 1

1. Do you think that the diagonals of every rectangle are perpendicular? If so, give an argument that supports your answer. If not, give an example that shows that the diagonals are not perpendicular. Explain your reasoning.

Take Note

Because rectangles and squares are parallelograms, the properties of parallelograms are true for these quadrilaterals as well.

Because squares are rhombi, the properties of rhombi are true for squares as well.

2. Do you think that the diagonals of every square are perpendicular? If so, give an argument that supports your answer. If not, give an example that shows that the diagonals are not perpendicular.

Summary Properties of Quadrilaterals

In this chapter, you have learned the following properties of quadrilaterals.

- A **parallelogram** is a quadrilateral in which the opposite sides are parallel.

  The opposite sides of a parallelogram are congruent.

  The opposite angles of a parallelogram are congruent.

  The consecutive angles of a parallelogram are supplementary.

  The diagonals of a parallelogram bisect each other.

- A **rhombus** is a parallelogram with four congruent sides.

  A diagonal of a rhombus bisects a pair of opposite angles.

  The diagonals of a rhombus are perpendicular.

- A **rectangle** is a parallelogram in which the angles are all right angles.

  The diagonals of a rectangle are congruent.

- A **square** is a rectangle in which all four sides are congruent.

  The diagonals of a square are perpendicular.

- A **kite** is a quadrilateral in which two pairs of adjacent sides are congruent but the opposite sides are not congruent.

  In a kite, only one pair of opposite angles are congruent.

  In a kite, the diagonal that joins the vertices with the congruent angles is bisected by the other diagonal.

  The diagonals of a kite are perpendicular.

- A **trapezoid** is a quadrilateral in which exactly one pair of opposite sides is congruent.

- An **isosceles trapezoid** is a trapezoid in which the nonparallel sides are congruent.

  The base angles of an isosceles trapezoid are congruent.

  The diagonals of an isosceles trapezoid are congruent.

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6.5 Stained Glass

Sum of the Interior Angle Measures in a Polygon

Objectives

In this lesson, you will:

- Determine the sum of the interior angle measures in a convex polygon.
- Determine the measure of an interior angle of a regular polygon.
- Determine the number of sides in a regular polygon given the measure of an interior angle.

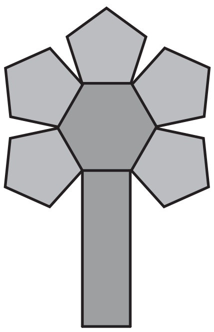
Key Terms

- interior angle
- convex polygon
- regular polygon

Modern stained glass artwork and windows are created by cutting out pieces of glass and fitting them together with a metal strip that is grooved to hold the glass. All the metal strips that hold pieces of glass in a window or artwork are "glued" together by using molten metal.

Problem 1 Stained Glass Flowers

A stained glass design is shown.



A. Identify the different kinds of polygons that are in the stained glass design.

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B. Draw one diagonal in the quadrilateral. What kinds of polygons are formed by the diagonal and the quadrilateral?

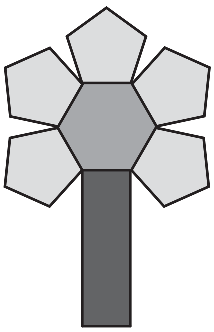
How many triangles are formed?

An **interior angle** is an angle that faces the inside of a polygon or shape, and is formed by consecutive sides of the polygon or shape. What is the sum of the interior angle measures of one triangle?

What is the sum of the interior angle measures of the rectangle? Explain how you determined your answer.

What is the sum of the measures of the interior angles of any quadrilateral?

C. Choose a pentagon from the stained glass design. Then choose one of the vertices from the pentagon and draw all of the diagonals that connect to this vertex.



How many triangles are formed by the diagonals?

What is the sum of the interior angle measures of one triangle?

What is the sum of the interior angle measures of any pentagon? Explain how you determined your answer.

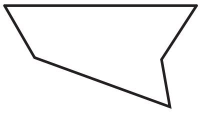
D. Choose the hexagon from the stained glass design. Then choose one of the vertices from the hexagon and draw all of the diagonals that connect to this vertex.

How many triangles are formed by the diagonals?

What is the sum of the interior angle measures of one triangle?

What is the sum of the interior angle measures of any hexagon? Explain how you determined your answer.

E. The polygons that you have been considering so far are *convex polygons*. We are concerned only with convex polygons in this lesson and the next lesson. A **convex polygon** is a polygon in which no segments can be drawn to connect any two vertices so that the segment is *outside* the polygon. The polygon on the left is a convex polygon. The polygon on the right is not a convex polygon. Draw the line segment on the polygon that shows that it is not a convex polygon.



Investigate Problem 1

1. Explain how the sum of the interior angle measures of a triangle can be used to determine the sum of the interior angle measures of any polygon.

2. How does the number of diagonals that connect to a single vertex of the polygon relate to the number of sides in a polygon?

   How does the number of triangles that are formed by drawing all of the diagonals that connect to a single vertex of the polygon relate to the number of sides in a polygon?

3. What is the sum of the measures of the interior angles of a heptagon (seven-sided polygon)? Show all your work.

4. Write a formula that you can use to determine the sum of the interior angle measures of an *n*-gon. Explain your reasoning.

5. Use your formula to calculate the sum of the interior angle measures of a dodecagon (12-sided polygon). Show all your work.

6. Remember that a **regular polygon** is a polygon in which all sides are equal in length and all angles are equal in measure.

What is the measure of an interior angle of a regular pentagon? Explain how you determined your answer.

The measure of an interior angle of a regular polygon is 144°. How many sides does the regular polygon have? Show all your work.

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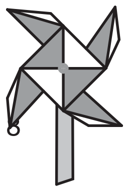
6.6 Pinwheels
Sum of the Exterior Angle Measures in a Polygon

Objective
In this lesson, you will:
- Determine the sum of the exterior angle measures in a polygon.

Key Term
- exterior angle

You've probably seen a pinwheel like the one shown. This pinwheel was made by using a square piece of paper. We will use our knowledge of polygons to create our own pinwheels that are more complicated.

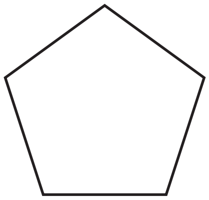


Problem 1 Making the Cut

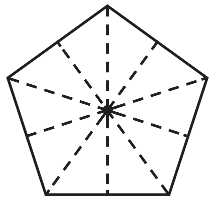
Your pinwheel will be made by using a piece of paper that is cut into the shape of a regular pentagon.

A. What is the measure of an interior angle in a regular pentagon?

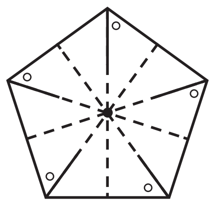
B. On a sheet of paper, use a protractor and ruler to draw the largest regular pentagon you can. Then cut out the pentagon.



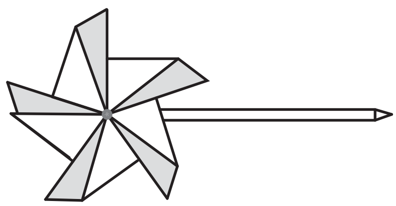
C. At each vertex, fold your pentagon so that the fold bisects the vertex angle, and then open the pentagon. Mark the point in the center of your pentagon where the folds meet.



D. Cut along the fold at each vertex about halfway to the center. At the upper left corner of each flap, use a hole punch to punch a hole.

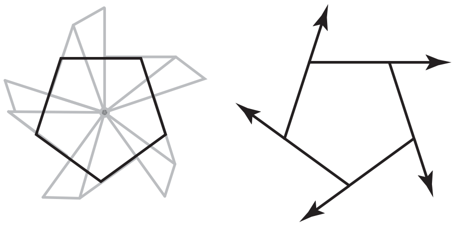


E. Carefully pull each corner with the hole toward the center of the pentagon. Then carefully put a push pin through the holes and then through the center of the polygon into the eraser head of your pencil. Your pinwheel is complete.



Investigate Problem 1

1. You may have noticed that when you were joining the flaps in the center, a pentagon similar to the one you started with can be seen. Look for the similar pentagon in your pinwheel.



As with triangles, you can consider the **exterior angles** of convex polygons. Whenever you extend one side at a vertex, you create an exterior angle that is acute, obtuse, or right. Number the exterior angles of the pentagon on the right.

If one exterior angle is drawn at each vertex, how many exterior angles are there for the regular pentagon?

What is $m\angle 1$? Explain your reasoning.

What is $m\angle 2$? Explain your reasoning.

What is $m\angle 3$? Explain your reasoning.

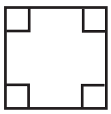
What is $m\angle 4$? Explain your reasoning.

6

What is $m\angle 5$? Explain your reasoning.

What is the sum of the measures of the exterior angles of the regular pentagon?

2. Extend each vertex of this square to create one exterior angle at each vertex.

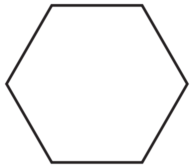


What is the measure of an interior angle of a square?

Determine the measure of each exterior angle. Explain how you determined your answers.

What is the sum of the measures of the exterior angles of a square?

3. Extend each vertex of this regular hexagon to create one exterior angle at each vertex.



What is the measure of an interior angle of a regular hexagon? Show all your work.

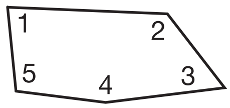
Determine the measure of each exterior angle. Explain how you determined your answers.

What is the sum of the measures of the exterior angles of a regular hexagon?

4. Without drawing a regular octagon, calculate the sum of the measures of the exterior angles of a regular octagon. Show all your work and explain how you calculated your answer.

5. Do you think that the sums you calculated in Question 1 to 4 are the same for any polygon, regular or not?

6. This pentagon is not regular. Extend each vertex of the pentagon to create one exterior angle at each vertex.



For each exterior angle, write an expression for its measure in terms of the measure of the adjacent interior angle.

Write the sum of your expressions and simplify the resulting expression.

What is the sum of the interior angle measures of a pentagon? Substitute this sum into the expression you wrote.

What is the sum of the measures of the exterior angles of any pentagon?

7. Consider any *n*-gon. Write an expression for the sum of the measures of the interior angles of the *n*-gon.

Complete the following expression for the sum of the measures of the exterior angles of the *n*-gon.

$n \cdot \underline{\qquad\quad} - (n - \underline{\quad\;}) \cdot 180^\circ$

Simplify the expression to calculate the sum of the measures of the exterior angles of any convex polygon.