CHAPTER

10 Problem Solving with Functions

Bricks have been used as a building material for almost 10,000 years. Bricklayers, or masons, are skilled tradesmen who build structures by laying down bricks and binding them together with mortar. You will use rational equations and functions to calculate how quickly teams of bricklayers can construct building foundations.

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10.1 Roots and Zeros
Calculating Roots of Quadratic Equations and Zeros of Quadratic Functions

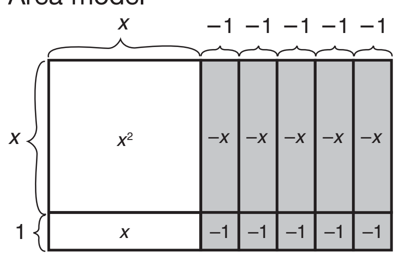
Objectives
In this lesson, you will:
- Solve quadratic equations.
- Calculate zeros of quadratic functions.

Key Terms
- Converse of the Multiplication Property of Zero
- roots
- zeros

Problem 1 Roots of Quadratic Equations

A quadratic expression of the form $x^2 + bx + c$ can be factored using an area model, a multiplication table, or the factors of the constant term c. The quadratic expression $x^2 - 4x - 5$ is factored using each method as shown.

- Area model



$x^2 - 4x - 5 = (x - 5)(x + 1)$

- Multiplication table

$\cdot$	x	-5
x	x^2	$-5x$
1	x	-5

$x^2 - 4x - 5 = (x - 5)(x + 1)$

- Factors of the constant term c

Factors of -5: $-5, 1$ $\quad -1, 5$

Sums: $-5 + 1 = -4$ $\qquad 5 + (-1) = 4$

$x^2 - 4x - 5 = (x - 5)(x + 1)$

The **Converse of the Multiplication Property of Zero** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

If $ab = 0$ *then* $a = 0$ *or* $b = 0$.

1. Factor and solve each quadratic equation.

a. $x^2 - 4x - 5 = 0$

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b. $x^2 - 8x + 12 = 0$

c. $x^2 - x - 6 = 0$

d. $x^2 - 5x - 24 = 0$

The solutions to quadratic equations are called **roots.**

2. Calculate the roots of each quadratic equation.

a. $x^2 - 16 = 0$

b. $x^2 - 5x + 6 = 0$

c. $x^2 - 5x - 36 = 0$

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3. Factor and solve each quadratic equation, if possible.

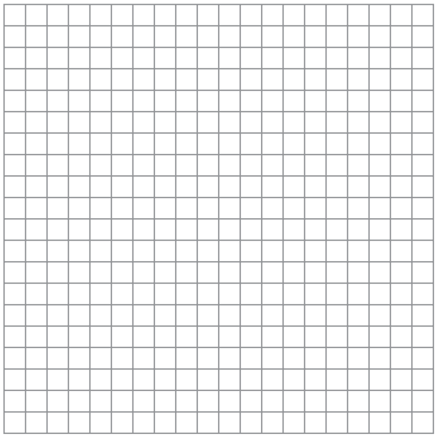
a. $x^2 + 10x - 75 = 0$

b. $x^2 - 10x + 25 = 0$

c. $x^2 - 11x = 0$

Problem 2 Zeros of Quadratic Functions

1. Graph the quadratic function $f(x) = x^2 - 4x - 5$ on the grid shown.



2. Identify the vertex, x- and y-intercepts, and line of symmetry. Label each on the grid.

a. Vertex:

b. y-intercept:

c. x-intercept(s):

d. Line of symmetry:

3. You calculated the roots of the quadratic equation $x^2 - 4x - 5 = 0$ in Problem 1 Question 1(a). What are the roots of $x^2 - 4x - 5 = 0$?

4. Compare the x-intercepts of the function $f(x) = x^2 - 4x - 5$ to the roots of the quadratic equation $x^2 - 4x - 5 = 0$. What do you notice?

5. The x-intercepts of a function are also called the **zeros** of the function. Why?

6. Calculate the zeros of each quadratic function, if possible.

a. $f(x) = x^2 - 4x + 3$

b. $f(x) = x^2 - 7x - 18$

c. $f(x) = x^2 - 11x + 12$

d. $f(x) = x^2 + 10x - 39$

Be prepared to share your solutions and methods.

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10.2 Poly High
Factoring Polynomials

Objectives
In this lesson, you will:
- Transform graphs of polynomial functions.
- Solve polynomial equations using factoring.

Key Terms
- polynomial equation
- greatest common factor
- factoring by grouping

Problem 1 Moving

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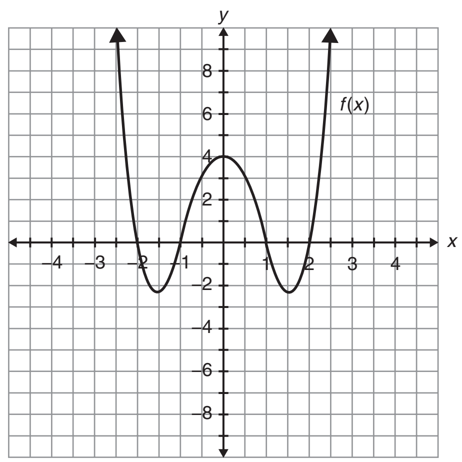
The graph of the polynomial function $f(x)$ is shown on the grid.



1. Identify the zeros, y-intercept, and line of symmetry for $f(x)$. Label each on the grid.

   a. Zeros:

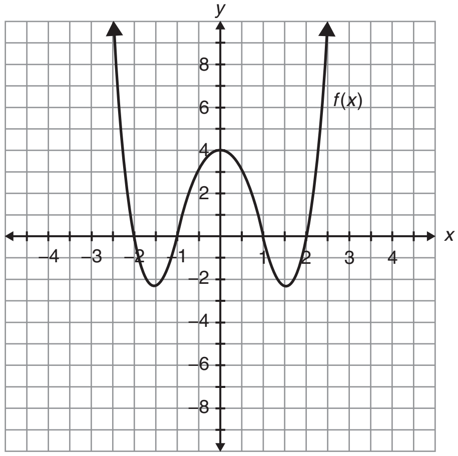
   b. y-intercept:

   c. Line of symmetry:

2. Sketch the graph of each transformed function.

a. $f(x) + 4$

b. $f(x + 2)$

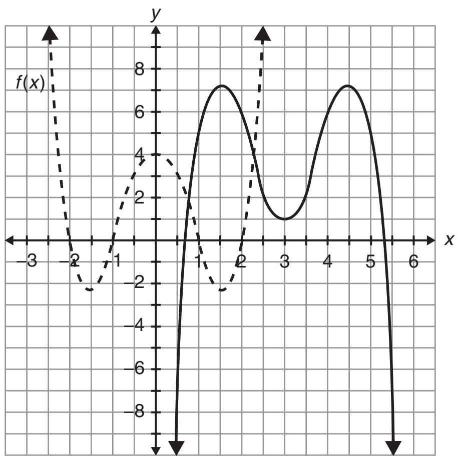


3. Describe how each graph is formed from the graph of $f(x)$.

a. Five is subtracted from $f(x)$: $f(x) - 5$

b. Four is subtracted from x: $f(x - 4)$

4. The graph of $f(x)$ is shown. Write a function in terms of $f(x)$ for the transformed graph.



Problem 2 Solving Polynomial Equations

A **polynomial equation** is an equation that can be written in the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0 = 0$, where a is any real number and n is a positive integer. A linear equation is a first-degree polynomial equation because the highest power is 1. A quadratic equation is a second-degree polynomial equation because the highest power is 2.

The Converse of the Multiplication Property of Zero can be used to solve polynomial equations if the polynomial is factorable.

1. Factor and solve each polynomial equation.

a. $x^2 - 5x + 4 = 0$

b. $x^4 - 5x^2 + 4 = 0$

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c. $x^4 - 8x^2 + 16 = 0$

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d. $x^4 + 10x^2 + 9 = 0$

The **greatest common factor** of a polynomial is the largest factor that is common to all terms of the polynomial. Solving some polynomial equations requires factoring the greatest common factor and then factoring the remaining factor.

For example, solve the polynomial equation $2x^3 - 8x = 0$.

$2x^3 - 8x = 0$

Greatest common factor: $2x$

$2x(x^2 - 4) = 0$

$$2x = 0 \quad \text{or} \quad x^2 - 4 = 0$$

$$x = 0 \quad \text{or} \quad (x - 2)(x + 2) = 0$$

$$x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -2$$

Check: $2(0)^3 - 8(0) = 0$

$2(2)^3 - 8(2) = 16 - 16 = 0$

$2(-2)^3 - e(-2) = -16 + 16 = 0$

2. Factor and solve each polynomial equation.

a. $4x^3 - 36x = 0$

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b. $4x^3 - 4x^2 - 24x = 0$

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c. $3x^3 - 27x^2 - 30x = 0$

Factoring by grouping is another method of factoring. To factor by grouping, create two groups of terms and factor the greatest common factor of each group. Then factor the greatest common factor of the groups.

For example, solve the polynomial equation $x^3 + 3x^2 - 4x - 12 = 0$.

$$x^3 + 3x^2 - 4x - 12 = 0$$

$$x^2(x + 3) - 4(x + 3) = 0$$

$$(x^2 - 4)(x + 3) = 0$$

$$x^2 - 4 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x = -3$$

$$x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = -3$$

Check: $(2)^3 + 3(2)^2 - 4(2) - 12 = 8 + 12 - 8 - 12 = 0$

$(-2)^3 + 3(-2)^2 - 4(-2) - 12 = -8 + 12 + 8 - 12 = 0$

$(-3)^3 + 3(-3)^2 - 4(-3) - 12 = -27 + 27 + 12 - 12 = 0$

3. Factor and solve each polynomial equation.

a. $x^3 + 4x^2 - 9x - 36 = 0$

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b. $x^3 - 5x^2 + 3x - 15 = 0$

c. $2x^4 + 4x^3 - 2x^2 - 4x = 0$

Be prepared to share your findings with the class.

10.3 Rational Thinking
Rational Equations and Functions

Objectives
In this lesson you will:
- Model situations with rational functions.
- Solve rational equations in one variable.

Key Terms
- rational equation
- extraneous solution

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Problem 1

One day, Dan Petersen was having a disagreement with his father. During their discussion, Mr. Petersen said, "You should listen to me. I know more. I've lived longer. You're not even half my age." That got Dan thinking about when he would be half of his father's age. Right now, Dan is 16 years old and his father is 36 years old.

1. Calculate Dan's age as a percentage of his father's age.

2. How old was Dan's father when Dan was born?

3. When will Dan be half of his father's age? How old will each be?

4. When will Dan be three quarters of his father's age? How old will each be?

5. Will Dan ever be as old as his father? Explain.

6. If x represents Dan's age, what expression represents his father's age?

7. What expression represents the ratio of Dan's age to his father's age?

8. What are the domain and range for this problem situation?

9. What are the domain and range of the mathematical function, $f(x) = \dfrac{x}{x + 20}$?

10. Are your answers for Questions 8 and 9 the same? Explain.

11. Complete the following table of values.

x	$\dfrac{x}{x + 20}$
−70	
−60	
−50	
−40	
−30	
−20	
−10	
0	

x	$\dfrac{x}{x + 20}$
10	
15	
20	
30	
40	
50	
60	
70	

12. Create a graph using the values from the table.

13. What are the x- and y-intercepts?

14. What are the intervals of increase and decrease?

15. Is the function continuous or discontinuous? If it is discontinuous, name any locations of discontinuity.

16. Describe the end behavior.

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17. What are the asymptotes?

18. Explain why the function has the asymptotes it does. Your explanation may be in terms of either the mathematical function or of the problem situation.

19. Describe the behavior near the vertical asymptote.

20. Is the situation in this problem an example of direct variation, inverse variation, or neither? Explain your answer.

21. What type of function models this situation?

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Problem 2

A **rational equation** is an equation containing one or more rational expressions. You have already solved simple rational equations with a single variable as a denominator.

1. Solve each of the following rational equations.

   a. $\frac{-3}{x} = -7$

   b. $\frac{5}{x} - 12 = 25$

   c. $\frac{5}{x} - 12 = \frac{25}{x}$

   d. $\frac{5}{x} = 25 + \frac{5}{x}$

   e. Does your solution check in part (d)? Explain

   f. When working with rational equations, you must limit your solution space based on the domain of the rational expressions. Explain.

In some rational equations, a rational expression is equal to a constant or to another rational expression. You can think of these equations as proportions. To solve them, you can cross multiply.

2. Solve each of the following rational equations. Make sure to list the restrictions to your solution set.

a. $\frac{12}{x + 5} = -2$

b. $\frac{7}{x + 3} = \frac{8}{x - 2}$

c. $\frac{x - 7}{x + 4} = -3$

d. $\frac{x - 5}{x - 2} = \frac{8}{9}$

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e. $\frac{x^2 - 5x}{4} = \frac{8x}{2}$

f. $\frac{3x^3 - 5x}{3} = \frac{2x^3 - 10x + 4}{2}$

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3. Solve the equation.

a. $\frac{5x}{6} - \frac{11}{12} + \frac{3x}{4} = \frac{2}{3}$

b. What was the first step in your solution path in part (a)? Why?

4. Using the same method, solve each of the following rational equations. Make sure to list the restrictions to your solution set.

a. $\frac{-7}{x} + \frac{4}{7} - \frac{8}{x} = \frac{9}{7}$

b. $\frac{3}{x + 1} + \frac{2}{5x + 5} = \frac{-3}{x + 1}$

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c. $\frac{3}{a - 1} + \frac{2}{5a + 5} = \frac{-3}{a^2 - 1}$

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d. $\frac{3}{y} + \frac{-4}{5y} = -11$

5. Solve the rational equation.

a. $\dfrac{x + 3}{x^2 - 1} + \dfrac{-2x}{x - 1} = 1$

b. What happens when you check your answers?

c. While solving an equation, you sometimes multiply by a variable and increase the degree of the equation, thereby increasing the number of solutions. This is what happened in part (a). The extra solution is called an **extraneous solution.** In part (a), what is the actual solution, and what is the extraneous solution?

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6. Solve each of the following rational equations. Make sure to identify any extraneous solutions, and make sure to list the restrictions to your solution set.

a. $\frac{3}{x+1} - \frac{1}{x-1} = \frac{2}{x^2-1}$

b. $\frac{-2}{x+3} + \frac{3}{x-2} = \frac{5}{x^2+x-6}$

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c. $\frac{1}{x+2} + 1 = \frac{8}{x^2 - 2x - 8}$

d. $\frac{1}{x-5} = \frac{5}{x^2 + 2x - 35}$

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10.4 Work, Mixture, and More
Applications of Rational Equations and Functions

Objectives
In this lesson, you will:
- Solve work problems.
- Solve mixture problems.
- Solve cost problems.

Key Terms
- work problems
- mixture problems
- cost problems

Problem 1 Work Problems

There is a class of problems that can be solved using rational equations and functions commonly called **work problems.**

1. Two teams of bricklayers are working on a new development of townhouses. Each quad of townhouses has the exact same block foundation. One team of bricklayers can complete this foundation in 30 hours, and the second team can complete it in 40 hours. If both teams work on the same foundation at once, how long will it take them to finish one set of quad townhouses? Complete parts (a) through (f) to answer this question.

a. How much of the foundation will the first team of bricklayers complete in 10 hours? 20 hours? 1 hour? *x* hours?

b. How long will it take the first team to finish $\frac{1}{2}$ of the job? $\frac{1}{5}$ of the job? $\frac{1}{30}$ of the job?

c. How much of the foundation will the second team of bricklayers complete in 10 hours? 20 hours? 1 hour? x hours?

d. How long will it take the second team to finish $\frac{1}{10}$ of the job? $\frac{1}{5}$ of the job? $\frac{1}{40}$ of the job?

e. If the two teams work together for x hours, write an expression that represents how much of the job they would complete.

f. Using the expression you wrote in part (e), write and solve an equation that represents one completed job. Then answer the following question: If both teams work on the same foundation at once, how long will it take them to finish one set of quad townhouses?

2. If Elaine is working alone, she can complete a job in 10 minutes. If she is working with José, the two of them can complete this job in 6 minutes. How long would it take José to complete this job if he is working alone? Complete parts (a) through (d) to answer this question.

a. Define a variable for the time it takes José to do the job alone.

b. Using the variable you defined in part (a), what fraction of the job can José complete in one minute?

c. What fraction of the job can Elaine complete in 6 minutes?

d. Write and solve an equation that represents one completed job when Elaine and José are working together. Use this equation to calculate the number of minutes it would take José to complete the job working alone.

Problem 2 Mixture Problems

Another class of problems using rational equations and functions is concentration, or **mixture problems.**

1. A saline or salt solution of 120 mL contains 10% salt. How much water would need to be added to this solution for it to contain only 2% salt? Complete parts (a) through (c) to answer this question.

   a. In the original saline solution, how many milliliters are there of salt? Of water? Explain how you know.

   b. What would the salt concentration of the solution be if you added 80 mL of water? 180 mL of water?

   c. Define a variable for the amount of water you need to add to the original solution. Use this variable to write and solve an equation to find how much water needs to be added to make the solution a 2% solution. Then answer the following question: How much water would need to be added to this solution for it to contain only 2% salt?

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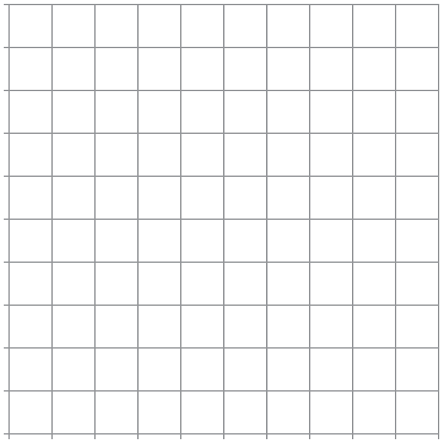
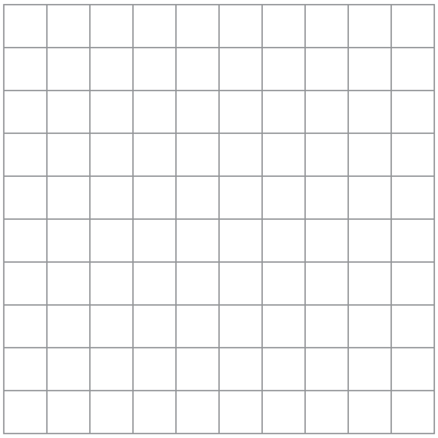
2. For the mixture described in Problem 2, define a function $C(x)$ for the concentration, where x is the amount of water added.

a. Using $C(x)$, graph the problem situation. Then identify the domain and range of the graph.



b. Now graph $C(x)$ as a mathematical function. Determine the domain, range, asymptotes, discontinuities, and end behavior of this function.



3. A 20% sulfuric acid 20 mL solution will be mixed with a 5% solution to produce other concentration solutions.

a. Will the new solution ever have a concentration of 20%? 5%? Explain.

b. If 20 mL of 20% solution is mixed with 10 mL of 5% solution, what will be the concentration of the resulting solution? Explain.

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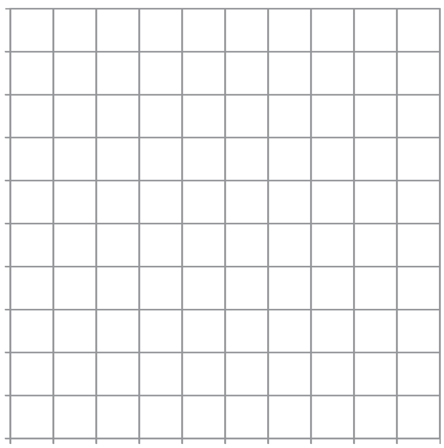
c. Define a variable for the amount of 5% solution added, and then define a function $S(x)$ for the concentration of the resulting solution.

d. Using $S(x)$, graph the problem situation. Then identify the domain and range of the graph.



e. Now graph $S(x)$ as a mathematical function. Determine the domain, range, asymptotes, discontinuities, and end behavior of this function.

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Problem 3 Cost of Ownership Problems

Another class of problems has to do with the total cost of owning something over time. These problems are called **cost problems.**

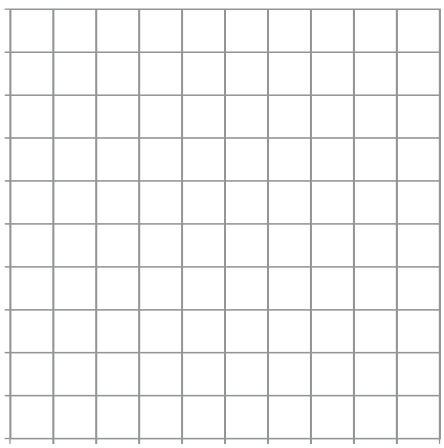
1. A new High Definition television costs \$1600 and uses about \$15 of electricity a year.

a. Assuming that this television is reliable and its only costs of ownership are the original cost and the cost of electricity, what would be the average annual cost of owning this television after 5 years? 10 years?

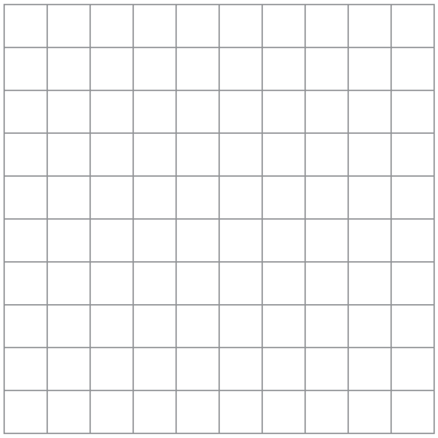
b. After how many years would the average annual cost be $115?

c. Determine the function of the cost of ownership of this television and define the variables.

d. Using $C(t)$, graph the problem situation. Then identify the domain and range of the graph.



e. Now graph $C(t)$ as a mathematical function. Determine the domain, range, asymptotes, discontinuities, and end behavior of this function.



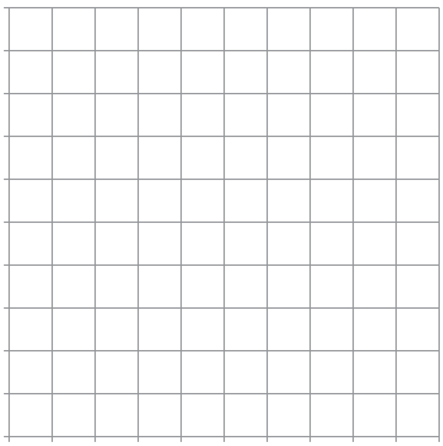
2. A new luxury automobile costs $75,000 to purchase, and it is estimated to cost about $8500 a year to own including fuel, service, repairs, and insurance.

a. Assuming that these estimates are reliable and will not change, what would be the average annual cost of ownership after 5 years? 10 years?

b. After how many years would the average annual cost be $10,000?

c. Determine the function of the cost of ownership of this automobile and define the variables.

d. Using $C(t)$, graph the problem situation. Then identify the domain and range of the graph.



e. Now graph $C(t)$ as a mathematical function. Determine the domain, range, asymptotes, discontinuities, and end behavior of this function.

Be prepared to share your work with another pair, group, or the entire class.

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10.5 Rad Man!
Radical Equations and Functions

Objectives

In this lesson, you will:

- Transform graphs of radical functions.
- Solve equations involving radicals.

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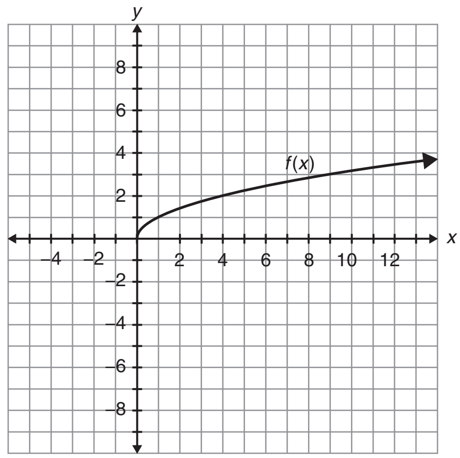
Problem 1 Moving

The graph of the basic radical function $f(x) = \sqrt{x}$ is shown on the grid.



1. Identify the zeros and y-intercept of $f(x) = \sqrt{x}$. Label each point on the grid.

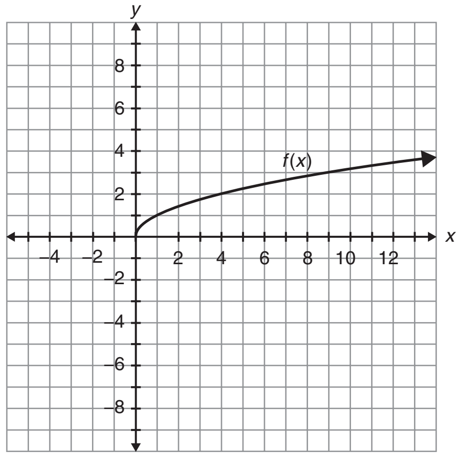
a. Zeros:

b. y-intercept:

2. Sketch the graph of each transformed function.

   a. $f(x) - 4$

   b. $f(x - 2)$

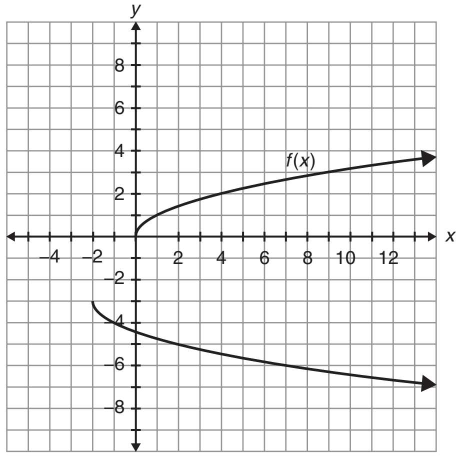


3. Describe how each graph is formed from the graph of $f(x)$.

   a. Five is added to $f(x)$: $f(x) + 5$

   b. Four is added to x: $f(x + 4)$

4. The graph of $f(x)$ is shown. Write a function in terms of $f(x)$ for the transformed graph.



Problem 2

To solve an equation that contains a variable within a radical, you must first isolate the radical on one side of the equation. Once you have done this, remove the radical by raising both sides to the index of the radical.

For example: $\sqrt{2x} - 5 = 2$

$$\sqrt{2x} = 7$$

$$(\sqrt{2x})^2 = 7^2$$

$$2x = 49$$

$$x = \frac{49}{2}$$

Check: $\sqrt{2\left(\frac{49}{2}\right)} - 5 = 2$

1. Apply these steps to solve the following equation.

   $\sqrt{3x - 4} = 5$

2. Solve each equation.

   a. $\sqrt{x} = 6$

   b. $\sqrt{2x} = 3$

   c. $\sqrt[3]{2x - 3} = 2$

   d. $4\sqrt{x - 6} = 8$

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e. $\sqrt{2x + 1} = 5$

f. $2\sqrt[3]{x} + 16 = 0$

g. $\sqrt[4]{3x - 1} + 9 = 8$

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Problem 3

In general, when solving for a variable in an equation that is raised to the *n*th power, you isolate the variable on one side of the equation and then take the *n*th root of each side (square root, cube root, fourth root, or other appropriate root).

In other words, to eliminate the *n*th power of the variable, you apply the *inverse operation*, which means taking the *n*th root of each side of the equation.

When solving any type of equation, you should always check to see whether the solution you obtain actually solves the original equation.

This is particularly important when you solve equations involving radicals because, in the process of solving the equation, apparent solutions may be introduced that do *not* solve the original equation. These apparent solutions are called extraneous solutions.

For example, whenever both sides of an equation are raised to a power, new solutions may be introduced that are not solutions to the original equation. Raising the degree of an equation increases the number of solutions.

Starting with $x = 2$, let's work backwards to an equation for this solution.

$x = 2$	Start with the original equation.	
$x^2 = 4$	Square both sides of the equation.	
$x^2 - 4 = 4 - 4$	Subtract 4 from each side.	
$x^2 - 4 = 0$	Solve the resulting quadratic equation.	
$(x - 2)(x + 2) = 0$		
$x - 2 = 0$	or	$x + 2 = 0$
$x = 2$	or	$x = -2$

There are two solutions to the quadratic equation, but one ($x = -2$) is not a solution to the original equation. This extraneous solution results because the degree of the original equation is increased. The degree of the original equation is 1 (a linear equation). When you square each side of the equation, the degree is increased to 2 (a quadratic equation).

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1. Solve each of the following and check for extraneous solutions or roots.

   a. $x - \sqrt{x} = 2$

   b. $x - 1 = \sqrt{x + 1}$

c. $-x + \sqrt[3]{3x^2 + 3x} = 1$

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Be prepared to share your methods and solutions.

10.6 Connections
Algebraic and Graphical Connections

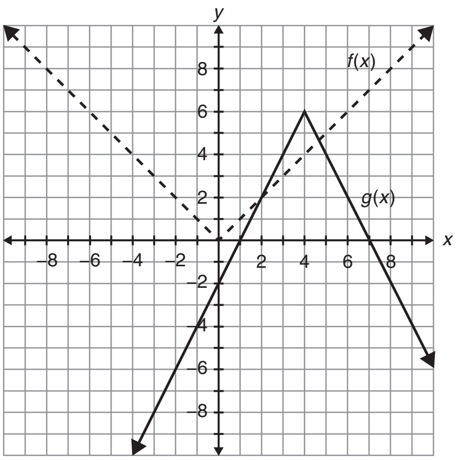
Objectives

In this lesson, you will:

- Describe transformations algebraically and graphically.
- Know the relationship between the zeros of a function and the roots of an equation.
- Solve equations algebraically and graphically.

Problem 1 Transformations

The graph of the basic absolute value function $f(x) = |x|$ and a transformed absolute value function $g(x)$ are shown on the grid.



1. Identify each characteristic for $f(x)$.

   a. x-intercept(s):

   b. y-intercept(s):

   c. Vertex:

   d. Line of symmetry:

2. Identify each characteristic for $g(x)$.

   a. x-intercept(s):

   b. y-intercept(s):

   c. Vertex:

   d. Line of symmetry:

3. Describe how the graph of $g(x)$ is formed from the graph of $f(x)$ using four transformations.

4. Write a function in terms of $f(x)$ for $g(x)$.

5. Write a function for $g(x)$ using absolute value.

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6. Is $f(x)$ an even function? An odd function? Explain.

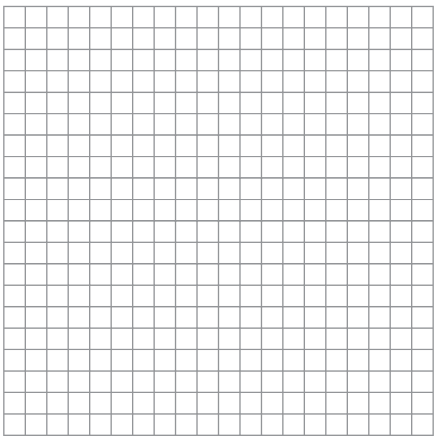
7. Is $g(x)$ an even function? An odd function? Explain.

Problem 2 Roots and Zeros

1. Describe the relationship between the zeros of the function $f(x) = y$ and the roots of the equation $y = 0$.

2. Calculate the roots of the equation $x^3 - 4x^2 - 4x + 16 = 0$ algebraically.

3. Graph the function $f(x) = x^3 - 4x^2 - 4x + 16$ on the grid shown.



4. Identify the intercepts for $f(x)$.

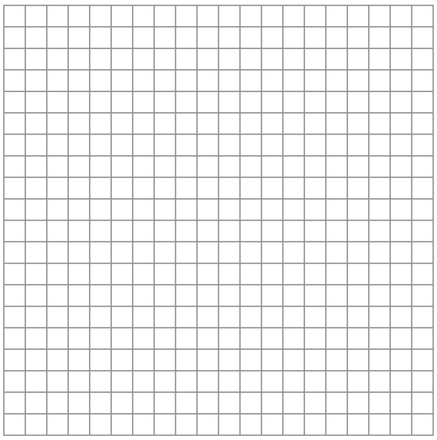
a. x-intercept(s):

b. y-intercept(s):

5. What do you notice about the roots of the equation from Question 2 and the *x*-intercepts of the function from Question 4?

6. Calculate the roots of the equation $x^3 - 4x^2 - 4x = 0$ algebraically.

7. Graph the function $g(x) = x^3 - 4x^2 - 4x$ on the grid shown.



8. Identify the intercepts for $g(x)$.

a. *x*-intercept(s):

b. *y*-intercept(s):

9. Calculate the x-intercepts for $g(x) = x^3 - 4x^2 - 4x$ using a graphing calculator by performing the following.

   a. Press the Y= button. Enter the function as Y_1.

   b. Graph the function using appropriate bounds.

   c. Press the 2ND button and the TRACE button. You will see the CALC menu.

   d. Select 2: Zero.

   e. You will be prompted for the left bound. Use the left and right buttons to trace along the curve. Move the cursor to a point to the left of the x-intercept and press ENTER.

   f. You will be prompted for the right bound. Use the left and right buttons to trace along the curve. Move the cursor to a point to the right of the x-intercept and press ENTER.

   g. You will be prompted for a guess. Use the left and right buttons to trace along the curve. Move the cursor as close to the x-intercept as possible and press ENTER.

   h. The coordinates of the x-intercept will be displayed.

   i. Repeat steps (a) through (h) for each additional x-intercept.

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Problem 3 Solving Equations Algebraically and Graphically

1. Solve each equation algebraically.

   a. $x^2 - 7x + 6 = 0$

b. $|x - 7| - 9 = 0$

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c. $\sqrt{3x} - 6 = 0$

d. $2^x = 64$

e. $\dfrac{6x + 7}{x} = x$

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2. Solve each equation graphically.

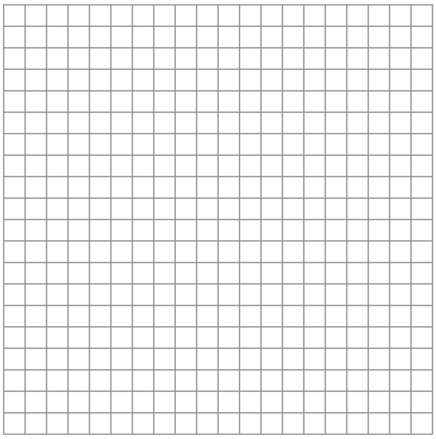
a. $\dfrac{x-2}{x} = x + 6$

Solution(s):

b. $-x^2 + 6 = 2^x$

Solution(s):

c. $|x + 1| = x^2 - 5$



Solution(s):

Be prepared to share your methods and solutions.

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